*Int. J. Solids Structures*, 1972, Vol. 8, pp. 999 to 1016. Pergamon Press. Printed in Great Britain

# STRESS ANALYSIS FOR A NONLINEAR VISCOELASTIC RUBBERLIKE MATERIAL

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**Abstract**—A nonlinear creep and a cylinder problem are studied based on the approximate constitutive equation introduced by Lianis. These constitutive equations have been shown to be suitable for the characterization of rubberlike materials under finite deformation. Numerical solutions, evaluated by using the experimental data for styrene butadiene rubber, are obtained by means of a finite-difference technique. The effects of nonlinearity of material properties are discussed.

## 1. INTRODUCTION

THE growing importance of plastics and high polymers as engineering materials has led to an increasing interest in the method of viscoelastic analysis. The linear analysis for infinitesimal deformations has received considerable attention in the past decade. However, recent investigation reveals that for many polymers, nonlinearity should be considered even at very small strains [1–15]. The necessity of characterizing and predicting the material responses leads to the recent development of nonlinear viscoelastic analysis.

In order to study the behavior of viscoelastic materials under large deformations, a number of investigators have used the principles of continuum mechanics to develop general constitutive equations. Such formulations usually involve the use of hereditary functionals of the deformation history. Coleman and Noll [16] have applied this approach to derive an approximation which applies to simple viscoelastic materials with fading memory under slow motions. It is known as the finite linear viscoelastic theory. The term "linear" refers to integrals, that is, only single integrals are shown in their equations. By a thermodynamic consideration, Lianis [17] showed that additional thermodynamic restrictions should be imposed on the constitutive equation. He also showed that, for an isotropic compressible material under isothermal conditions, there are nine independent material kernels. The number of kernel functions reduces to eight for incompressible materials. Other than these kernels, three steady-state coefficients should be determined to account for the equilibrium behavior.

Since three material coefficients and nine kernel functions are still beyond the realm of experimental evaluation, some seemingly reasonable approximations have to be introduced. In view of the Mooney–Rivlin approximation in rubber elasticity, Lianis [18] suggested a possible simplification analogous to Mooney–Rivlin materials for isotropic incompressible materials. His constitutive equation contains only four kernel functions and three steady-state coefficients. Intuitively, Lianis' approximation can be regarded as

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an extension of the Mooney–Rivlin material where the linear deformation history has been taken into account. A series of time varying tests were conducted by Lianis *et al.* [19–25]. It reveals that Lianis' approximation is convenient for characterizing various rubberlike viscoelastic materials over a wide range of deformation histories and time periods. However, the usefulness of such theory for engineering design and stress analysis should also depend on the feasibility of solving boundary value problems. It is therefore the purpose of the present investigation to analyze some engineering problems based on these approximate equations.

In Section 2, basic descriptions and a review of the approximate constitutive equations are given. In Section 3, uniaxial creep under constant load is analyzed. Numerical solutions are obtained by using the experimental results of uniaxial tensile relaxation tests on styrene butadiene rubber conducted by Goldberg [26].

In Section 4, a cylinder problem is analyzed. We show that the problem can be reduced to a single integral equation of Volterra type. Numerical solutions for a pressurized hollow cylinder are obtained by using a finite-difference technique. It shows that Lianis' approximation is convenient for solving stress analysis problems.

## 2. BASIC EQUATIONS

Consider a rectangular Cartesian coordinate system fixed in space. Let $\mathbf{X}$ be the coordinates of a material point of a body in its undeformed state with respect to the fixed system; let $\mathbf{x}$ represent the coordinates at time $\tau$ of the material point. Then,

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, \tau). \tag{2.1}$$

We shall use the following notations in matrix form

$$\mathbf{F}(\tau) = \left[\frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{X}}\right]; \qquad \mathbf{F} = \mathbf{F}(t) \tag{2.2}$$

for the deformation gradient,

$$\mathbf{F}_t(\tau) = \left[\frac{\partial \mathbf{x}(\tau)}{\partial \mathbf{x}(t)}\right] = \mathbf{F}(\tau)\mathbf{F}^{-1} \tag{2.3}$$

for the relative deformation gradient, where $t$ indicates the present time. The left and right Cauchy–Green tensors are, respectively

$$\mathbf{B}(\tau) = \mathbf{F}(\tau)\mathbf{F}^T(\tau), \qquad \mathbf{B}_t(\tau) = \mathbf{F}_t(\tau)\mathbf{F}_t^T(\tau) \tag{2.4}$$

$$\mathbf{C}(\tau) = \mathbf{F}^T(\tau)\mathbf{F}(\tau), \qquad \mathbf{C}_t(\tau) = \mathbf{F}_t^T(\tau)\mathbf{F}_t(\tau). \tag{2.5}$$

If the material is assumed to be incompressible, the condition is

$$\det|\mathbf{F}(\tau)| = 1 \qquad 0 \le \tau \le t. \tag{2.6}$$

The stress–strain relations of an isotropic viscoelastic incompressible material have been developed by Coleman and Noll [16]. Their formulation can be reduced to the form

$$\begin{aligned}\boldsymbol{\sigma}(t) = & -p(t)\mathbf{I}+(\phi+I_1\psi)\mathbf{B}-\psi\mathbf{B}^2+\int_{-\infty}^{t}[{}^0\phi_1(t-\tau)\mathbf{I}+{}^1\phi_1(t-\tau)\mathbf{B}+{}^2\phi_1(t-\tau)\mathbf{B}^2]\dot{\mathbf{C}}_t(\tau)\,\mathrm{d}\tau \\ & +\int_{-\infty}^{t}\dot{\mathbf{C}}_t(\tau)[{}^0\phi_1(t-\tau)\mathbf{I}+{}^1\phi_1(t-\tau)\mathbf{B}+{}^2\phi_1(t-\tau)\mathbf{B}^2]\,\mathrm{d}\tau \\ & +\mathbf{I}\int_{-\infty}^{t}\mathrm{tr}\{\dot{\mathbf{C}}_t(\tau)[{}^0\phi_2(t-\tau)\mathbf{I}+{}^1\phi_2(t-\tau)\mathbf{B}+{}^2\phi_2(t-\tau)\mathbf{B}^2]\}\,\mathrm{d}\tau \\ & +\mathbf{B}\int_{-\infty}^{t}\mathrm{tr}\{\dot{\mathbf{C}}_t(\tau)[{}^0\phi_3(t-\tau)\mathbf{I}+{}^1\phi_3(t-\tau)\mathbf{B}+{}^2\phi_3(t-\tau)\mathbf{B}^2]\}\,\mathrm{d}\tau \\ & +\mathbf{B}^2\int_{-\infty}^{t}\mathrm{tr}\{\dot{\mathbf{C}}_t(\tau)[{}^0\phi_4(t-\tau)\mathbf{I}+{}^1\phi_4(t-\tau)\mathbf{B}+{}^2\phi_4(t-\tau)\mathbf{B}^2]\}\,\mathrm{d}\tau \end{aligned} \tag{2.7}$$

where $\dot{\mathbf{C}}_t(\tau) = (\mathrm{d}/\mathrm{d}\tau)\mathbf{C}_t(\tau)$ at constant $\mathbf{X}$, $-p(t)$ is an arbitrary hydrostatic pressure, $\phi$ and $\psi$ are material functions of the first and second strain invariants $I_1 = \mathrm{tr}\,\mathbf{B}$ and $I_2 = \frac{1}{2}[I_1^2-\mathrm{tr}\,\mathbf{B}^2]$, ${}^i\phi_j(\tau)(i = 0,1,2; j = 1,2,3,4)$ are twelve relaxation functions of time and the strain invariants $I_1$ and $I_2$. ${}^i\phi_j$ satisfies

$$\lim_{t\to\infty} {}^i\phi_j(t) = 0. \tag{2.8}$$

Since $p(t)$ is an arbitrary function of $t$, the integral with kernels ${}^i\phi_2$ can be ignored without losing generality. By considering an analogous theory to the Mooney–Rivlin material in rubber elasticity, Lianis suggested the approximations

$$\phi \simeq a+b(I_1-3) \tag{2.9}$$

$$\psi \simeq c \tag{2.10}$$

where $a$, $b$, $c$ are material constants. The stress–strain relation is further simplified by assuming

$${}^0\phi_3 = {}^2\phi_3 = {}^0\phi_4 = {}^1\phi_4 = {}^2\phi_4 = 0. \tag{2.11}$$

The rest of relaxation functions are retermed by

$$\begin{aligned}\phi_0(t) &= {}^0\phi_1(t); \qquad \phi_1(t) = {}^1\phi_1(t); \\ \phi_2(t) &= {}^2\phi_1(t); \qquad \phi_3(t) = {}^1\phi_3(t).\end{aligned} \tag{2.12}$$

It is also assumed that these material functions $\phi_0$, $\phi_1$, $\phi_2$ and $\phi_3$ are independent of the strain invariants. The resulting form is then

$$\begin{aligned}\boldsymbol{\sigma}(t) = & -p(t)\mathbf{I}+[a+b(I_1-3)+cI_1]\mathbf{B}-c\mathbf{B}^2+2\int_{-\infty}^{t}\phi_0(t-\tau)\dot{\mathbf{C}}_t(\tau)\,\mathrm{d}\tau+\mathbf{B}\int_{-\infty}^{t}\phi_1(t-\tau)\dot{\mathbf{C}}_t(\tau)\,\mathrm{d}\tau \\ & +\int_{-\infty}^{t}\phi_1(t-\tau)\dot{\mathbf{C}}_t(\tau)\,\mathrm{d}\tau\mathbf{B}+\mathbf{B}^2\int_{-\infty}^{t}\phi_2(t-\tau)\dot{\mathbf{C}}_t(\tau)\,\mathrm{d}\tau \\ & +\int_{-\infty}^{t}\phi_2(t-\tau)\dot{\mathbf{C}}_t(\tau)\,\mathrm{d}\tau\mathbf{B}^2+\mathbf{B}\int_{-\infty}^{t}\phi_3(t-\tau)\dot{I}_1(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{2.13}$$

This simplified constitutive equation contains only four material kernel functions and three material constants. It has been shown that this equation is convenient on characterizing a large variety of rubber-like materials. For slow motions, the approximation (2.11) appears to be quite satisfactory [19–25].

## 3. UNIAXIAL CREEP

Consider a prismatic bar under uniaxial loading along its axis. The material of the bar is nonlinearly viscoelastic, incompressible and isotropic. Set the $x_1$-axis along the axis of the bar and $x_2$ and $x_3$ in the transverse directions. The deformation of the bar can be described by the relations:

$$
\begin{aligned}
x_1(\tau) &= \lambda(\tau) X_1 \\
x_2(\tau) &= \frac{1}{\lambda^{\frac{1}{2}}(\tau)} X_2 \\
x_3(\tau) &= \frac{1}{\lambda^{\frac{1}{2}}(\tau)} X_3
\end{aligned}
\tag{3.1}
$$

where $\lambda(\tau)$ is the stretch ratio and, in general, time dependent.

From equations (2.4) and (2.5), we have

$$
\mathbf{B} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix} \tag{3.2}
$$

$$
\mathbf{C}_t(\tau) = \begin{bmatrix} \frac{\lambda^2(\tau)}{\lambda^2} & 0 & 0 \\ 0 & \frac{\lambda}{\lambda(\tau)} & 0 \\ 0 & 0 & \frac{\lambda}{\lambda(\tau)} \end{bmatrix} \tag{3.3}
$$

$$
I_1(\tau) = \lambda^2(\tau) + \frac{2}{\lambda(\tau)} \tag{3.4}
$$

where $\lambda = \lambda(t)$.

Using the stress–strain relation (2.13), stress components $\sigma_1(t)$ and $\sigma_2(t)$ can be found. Since the specimen is stress free on its curved surface, $\sigma_2(t) = 0$, the hydrostatic pressure $p(t)$ can be deduced. The resulting expression for $\sigma_1(t)$ is then:

$$\sigma_1(t) = \left[a+b\left(\lambda^2+\frac{2}{\lambda}-3\right)+\frac{c}{\lambda}\right]\left(\lambda^2-\frac{1}{\lambda}\right)+2\int_{-\infty}^{t}\phi_0(t-\tau)\frac{d}{d\tau}\left[\frac{\lambda^2(\tau)}{\lambda^2}-\frac{\lambda}{\lambda(\tau)}\right]d\tau$$
$$+2\int_{-\infty}^{t}\phi_1(t-\tau)\frac{d}{d\tau}\left[\lambda^2(\tau)-\frac{1}{\lambda(\tau)}\right]d\tau+2\int_{-\infty}^{t}\phi_2(t-\tau)\frac{d}{d\tau}\left[\lambda^2\lambda^2(\tau)-\frac{1}{\lambda\lambda(\tau)}\right]d\tau$$
$$+\left(\lambda^2-\frac{1}{\lambda}\right)\int_{-\infty}^{t}\phi_3(t-\tau)\frac{d}{d\tau}\left[\lambda^2(\tau)+\frac{2}{\lambda(\tau)}\right]d\tau. \tag{3.5}$$

If the cross-sectional area of the bar in the undeformed state is $A_0$, at time $t$, the cross-sectional area becomes

$$A(t) = \frac{A_0}{\lambda}. \tag{3.6}$$

If the bar is undisturbed prior to time $t$ and subjected to a constant load $\pi$ for $t \geq 0^+$, then

$$\sigma_1(t) = \frac{\pi}{A_0}\lambda. \tag{3.7}$$

By substituting equation (3.7) into (3.5), we obtain an integral equation for determining $\lambda(t)$.

$$\frac{\pi}{A_0}\lambda = \left[a+2\phi_0(0)-3b+(b+\phi_3(0))\left(\lambda^2+\frac{2}{\lambda}\right)+\frac{c}{\lambda}+2\phi_2(0)\left(\lambda^2+\frac{1}{\lambda}\right)\right]\left(\lambda^2-\frac{1}{\lambda}\right)$$
$$-2\int_{0^+}^{t}\left\{\frac{\partial\phi_0}{\partial\tau}(t-\tau)\left[\frac{\lambda^2(\tau)}{\lambda^2}-\frac{\lambda}{\lambda(\tau)}\right]-\frac{\partial\phi_1}{\partial\tau}(t-\tau)\left[\lambda^2(\tau)-\frac{1}{\lambda(\tau)}\right]\right.$$
$$\left.-\frac{\partial\phi_2}{\partial\tau}(t-\tau)\left[\lambda^2\lambda^2(\tau)-\frac{1}{\lambda\lambda(\tau)}\right]-\frac{1}{2}\frac{\partial\phi_3}{\partial\tau}(t-\tau)\left(\lambda^2-\frac{1}{\lambda}\right)\left[\lambda^2(\tau)+\frac{2}{\lambda(\tau)}\right]\right\}d\tau. \tag{3.8}$$

In numerical calculation, the relaxation data for styrene butadiene rubber obtained by Goldberg [26] are used. The measurements were obtained from a series of uniaxial tensile relaxation tests at a constant temperature 0°C. From the tensile stress relaxation isochrones, the equilibrium coefficients were found to be:

$$a = 27\,\text{psi}, \qquad b = 0, \qquad c = 51{\cdot}6\,\text{psi}. \tag{3.9}$$

The relaxation functions $\phi_0$, $\phi_1$, $\phi_2$ and $\phi_3$ were calculated by solving simultaneous equations. The resulting data are listed in Table 1. Using the measured relaxation functions, the nonlinear integral equation (3.8) is solved by a method of finite difference. For each time step, the stretch ratio is determined by a trial-and-error method. Integrations are evaluated numerically by trapezoidal formulas. For convenience, a dimensionless time variable is introduced, $t/t_0$, where $t_0$ is chosen to be 100 sec. The stretch of prismatic bar is calculated for non-dimensional loading function $\pi/[\phi_0(0)A_0]$ which varies from 1·0 to 12·0. The creep curves and load–stretch curves are shown in Figs. 1 and 2, respectively. The load–stretch curves are concave to the stretch ratio axis and nonlinearity increases

TABLE 1. RELAXATION FUNCTIONS FOR STYRENE BUTADIENE RUBBER AT 0°C (GOLDBERG [26])

| Time (sec) | $\phi_0$ (psi) | $\phi_1$ (psi) | $\phi_2$ (psi) | $\phi_3$ (psi) |
|---|---|---|---|---|
| 0 | 12·30 | 9·00 | −4·50 | 9·00 |
| 1 | 12·30 | 9·00 | −4·50 | 9·00 |
| 2 | 9·70 | 8·35 | −4·28 | 8·35 |
| 4 | 7·43 | 7·65 | −3·83 | 7·65 |
| 6 | 6·35 | 7·26 | −3·63 | 7·26 |
| 10 | 5·07 | 6·87 | −3·44 | 6·87 |
| 15 | 4·12 | 6·51 | −3·26 | 6·51 |
| 20 | 3·51 | 6·29 | −3·15 | 6·29 |
| 30 | 2·68 | 5·93 | −2·97 | 5·93 |
| 40 | 2·18 | 5·60 | −2·80 | 5·60 |
| 50 | 1·84 | 5·32 | −2·66 | 5·32 |
| 60 | 1·62 | 5·04 | −2·52 | 5·04 |
| 120 | 0·96 | 3·92 | −1·96 | 3·92 |
| 180 | 0·66 | 3·04 | −1·52 | 3·04 |
| 360 | 0·25 | 1·64 | −0·82 | 1·64 |
| 540 | 0·09 | 0·70 | −0·35 | 0·70 |
| 720 | 0 | 0 | 0 | 0 |

sharply with time. The creep curves also show that the material is essentially nonlinearly elastic for small stretch ratios and time effects become increasingly important as the deformation increases.

## 4. CYLINDER PROBLEM

The determination of stress distributions in a viscoelastic cylinder has been one of the most extensively studied topics in the field of viscoelasticity. Solutions for cylinders made of linear viscoelastic materials have been attempted by many investigators [27–29]. The problems for nonlinearly viscoelastic materials under small finite deformation has been analyzed by Huang and Lee [30] and Ting [31, 32]. In their analyses, a stress–strain relation in multiple-integral form was assumed. Numerical solutions were obtained by considering the approximations for short-time ranges. The strains induced were less than 5 per cent. In this paper, we consider a different type of stress–strain relation which is suitable for characterizing rubber-like materials. It allows much larger deformations. Since the stress–strain relation contains only single integrals, the numerical calculations are much simpler than those involved in multiple-integral formulation.

Consider an infinitely long cylindrical tube made of a homogeneous, isotropic, incompressible nonlinearly viscoelastic material (Fig. 3). The loading is limited to axially symmetric pressures on the inner and outer boundaries. Any material particle with radial coordinate $R$ in the undeformed state moves to the radial coordinate $r(R, t)$ in the deformed state. We shall consider a plane strain problem where the deformation is described by the following relations:

$$\begin{aligned} x_1 &= \frac{r}{R}X_1 = f(R)X_1 \\ x_2 &= \frac{r}{R}X_2 = f(R)X_2 \qquad (4.1) \\ x_3 &= X_3. \end{aligned}$$

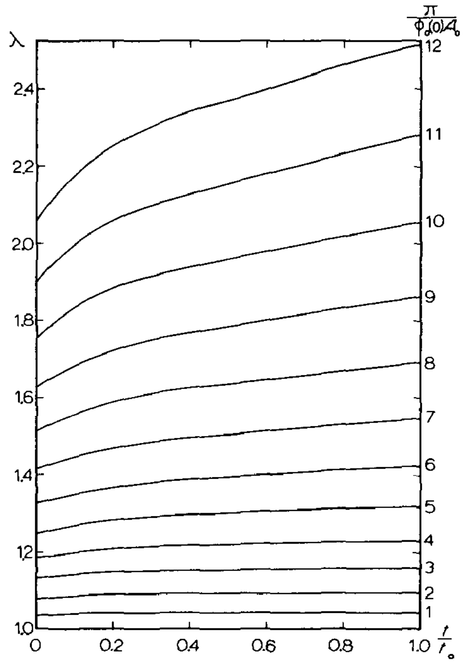


FIG. 1. Uniaxial creep curves under constant loading.

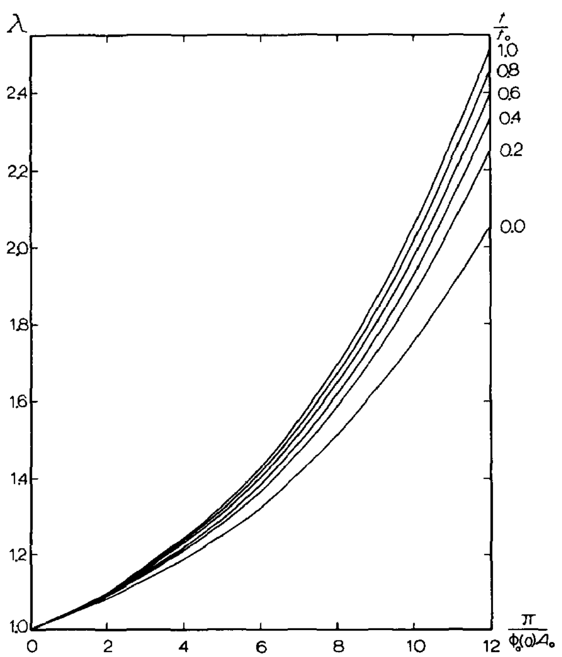


FIG. 2. Load–stretch ratio curves for uniaxial creep.

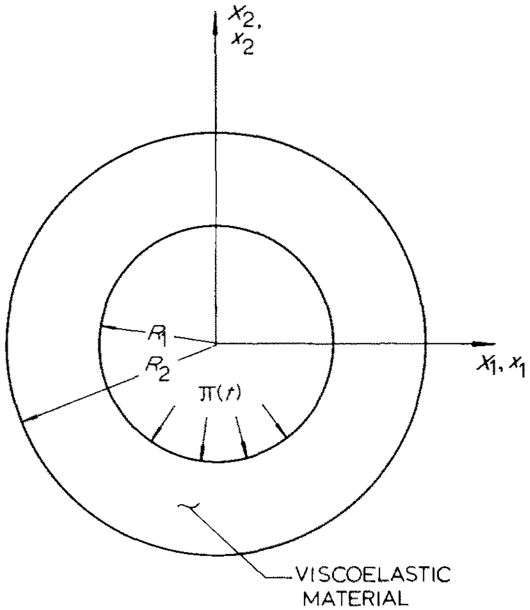


FIG. 3. Geometry of a viscoelastic cylinder.

The condition of incompressibility requires that

$$\pi[r^2 - r_1^2(t)] = \pi[R^2 - R_1^2] \tag{4.2}$$

where $r_1(t)$ and $R_1$ are, respectively, the inner radii of the deformed state and undeformed state. Denoting

$$\beta(\tau) = r_1^2(\tau) - R_1^2, \qquad \beta = \beta(t) \tag{4.3}$$

then,

$$r^2(\tau) = R^2 + \beta(\tau), \qquad r = r(t) \tag{4.4}$$

or

$$f^2(\tau) = 1 + \frac{\beta(\tau)}{R^2}, \qquad f = f(t). \tag{4.5}$$

Without losing generality, we choose a typical material point with particular coordinate

$$X_1 = R \quad \text{and} \quad X_2 = 0. \tag{4.6}$$

For this particular material particle, the deformation gradient $\mathbf{F}$ is given in matrix form by

$$\mathbf{F} = \begin{bmatrix} f + \frac{\partial f}{\partial R} R & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.7}$$

In view of the incompressibility condition, we get

$$f+\frac{\partial f}{\partial R}R = \frac{1}{f}. \tag{4.8}$$

Therefore,

$$\mathbf{F} = \begin{bmatrix} \frac{1}{f} & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.9}$$

The relative deformation gradient $\mathbf{F}_t(\tau)$ has the form

$$\mathbf{F}_t(\tau) = \mathbf{F}(\tau)\mathbf{F}^{-1} = \begin{bmatrix} \frac{f}{f(\tau)} & 0 & 0 \\ 0 & \frac{f(\tau)}{f} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.10}$$

The corresponding left and right Cauchy–Green tensors are

$$\mathbf{B} = \mathbf{C} = \begin{bmatrix} \frac{1}{f^2} & 0 & 0 \\ 0 & f^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.11}$$

and

$$C_t(\tau) = \begin{vmatrix} \frac{f^2}{f^2(\tau)} & 0 & 0 \\ 0 & \frac{f^2(\tau)}{f^2} & 0 \\ 0 & 0 & 0 \end{vmatrix}. \tag{4.12}$$

The strain invariants are

$$I_1 = I_2 = f^2+\frac{1}{f^2}+1. \tag{4.13}$$

The nonvanishing physical stress components are: the radial stress $\sigma_r$, the circumferential stress $\sigma_\theta$ and the axial stress $\sigma_z$. They have the form

$$\begin{aligned}\sigma_{r,\theta,z} = {} & -p(t)+[a+b(I_1-3)+cI_1]B_{r,\theta,z}-cB^2_{r,\theta,z} \\ & +2\int_{0^-}^{t} \phi_0(t-\tau)[\dot{C}_t(\tau)]_{r,\theta}\,\mathrm{d}\tau+2B_{r,\theta}\int_{0^-}^{t} \phi_1(t-\tau)[\dot{C}_t(\tau)]_{r,\theta}\,\mathrm{d}\tau \\ & +2B^2_{r,\theta}\int_{0^-}^{t} \phi_2(t-\tau)[\dot{C}_t(\tau)]_{r,\theta}\,\mathrm{d}\tau+B_{r,\theta,z}\int_{0^-}^{t} \phi_3(t-\tau)\dot{I}_1(\tau)\,\mathrm{d}\tau\end{aligned} \tag{4.14}$$

where the material is assumed to be undisturbed prior to $t = 0$. Specifically in terms of the unknown function $f$, the stress components $\sigma_r$ and $\sigma_\theta$ have the form

$$\begin{aligned}\sigma_r = & -p(t)+\left[a+b\left(f^2+\frac{1}{f^2}-2\right)+c\left(f^2+\frac{1}{f^2}+1\right)\right]\frac{1}{f^2}-c\frac{1}{f^4}\\ & +2\int_{0^-}^{t}\left[f^2\phi_0(t-\tau)+\phi_1(t-\tau)+\frac{1}{f^2}\phi_2(t-\tau)\right]\frac{\partial}{\partial\tau}\left[\frac{1}{f^2(\tau)}\right]\mathrm{d}\tau\\ & +\frac{1}{f^2}\int_{0^-}^{t}\phi_3(t-\tau)\frac{\partial}{\partial\tau}\left[f^2(\tau)+\frac{1}{f^2(\tau)}+1\right]\mathrm{d}\tau \end{aligned} \tag{4.15}$$

$$\begin{aligned}\sigma_\theta = & -p(t)+\left[a+b\left(f^2+\frac{1}{f^2}-2\right)+c\left(f^2+\frac{1}{f^2}+1\right)\right]f^2-cf^4\\ & +2\int_{0^-}^{t}\left[\frac{1}{f^2}\phi_0(t-\tau)+\phi_1(t-\tau)+f^2\phi_2(t-\tau)\right]\frac{\partial}{\partial\tau}[f^2(\tau)]\,\mathrm{d}\tau\\ & +f^2\int_{0^-}^{t}\phi_3(t-\tau)\frac{\partial}{\partial\tau}\left[f^2(\tau)+\frac{1}{f^2(\tau)}+1\right]\mathrm{d}\tau \end{aligned} \tag{4.16}$$

where the specific form of $f$ is given in equation (4.5). The equations of equilibrium have the form

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}(\sigma_r-\sigma_\theta) = 0 \tag{4.17}$$

$$\frac{\partial p}{\partial\theta} = \frac{\partial p}{\partial z} = 0. \tag{4.18}$$

Equation (4.18) implies

$$p = p(r, t)$$

and hence, in terms of the undeformed coordinates

$$p = \bar{p}(R, t). \tag{4.19}$$

It is convenient to use the undeformed coordinates for the present analysis. In view of the relations

$$r = fR, \qquad \frac{\partial r}{\partial R} = \frac{\partial f}{\partial R}R+f = \frac{1}{f} \tag{4.20}$$

equation (4.17) becomes

$$\frac{\partial\sigma_r}{\partial R}+\frac{\sigma_r-\sigma_\theta}{f^2R} = 0. \tag{4.21}$$

Since

$$\frac{1}{f^2} = \frac{R^2}{R^2+\beta}$$

we have

$$\sigma_r = -\int^R \frac{(\sigma_r - \sigma_\theta)R\,dR}{R^2+\beta} + \alpha(t) \tag{4.22}$$

where $\alpha(t)$ is an arbitrary function of time $t$. To obtain the expression of $\sigma_r - \sigma_\theta$, we use the relations (4.15) and (4.16) which give

$$\begin{aligned}\sigma_r - \sigma_\theta = {} & \left[a + b\left(\frac{\beta}{R^2} - \frac{\beta}{R^2+\beta}\right) + c\right]\left[\frac{R^2}{R^2+\beta} - \frac{R^2+\beta}{R^2}\right] \\ & + 2\int_{0^-}^t \phi_0(t-\tau)\frac{\partial}{\partial\tau}\left[\frac{R^2+\beta}{R^2+\beta(\tau)} - \frac{R^2+\beta(\tau)}{R^2+\beta}\right] d\tau \\ & + 2\int_{0^-}^t \phi_1(t-\tau)\frac{\partial}{\partial\tau}\left[\frac{R^2}{R^2+\beta(\tau)} - \frac{R^2+\beta(\tau)}{R^2}\right] d\tau \\ & + 2\int_{0^-}^t \phi_2(t-\tau)\frac{\partial}{\partial\tau}\left[\frac{R^4}{(R^2+\beta)(R^2+\beta(\tau))} - \frac{(R^2+\beta)(R^2+\beta(\tau))}{R^4}\right] d\tau \\ & + \left(\frac{R^2}{R^2+\beta} - \frac{R^2+\beta}{R^2}\right)\int_{0^-}^t \phi_3(t-\tau)\frac{\partial}{\partial\tau}\left[\frac{R^2+\beta(\tau)}{R^2} + \frac{R^2}{R^2+\beta(\tau)} + 1\right] d\tau. \end{aligned} \tag{4.23}$$

By substituting equation (4.23) into (4.22), we find that equation (4.22) can be directly integrated with respect to the space variable $R$. The resulting relation for $\sigma_r$ has the form

$$\begin{aligned}\sigma_r = {} & \alpha(t) - \frac{1}{2}(a+c)\left[\frac{\beta}{R^2+\beta} + \log\frac{R^2+\beta}{R^2}\right] - \frac{b}{2}\left[\log\frac{R^2}{R^2+\beta} + \frac{\beta^2}{R^2(R^2+\beta)} - \frac{R^4}{2(R^2+\beta)^2}\right] \\ & - \int_{0^-}^t \left[\phi_0(t-\tau) - \frac{1}{2}\phi_3(t-\tau)\right]\frac{\partial}{\partial\tau}\left[\log\frac{R^2+\beta(\tau)}{R^2+\beta} + \frac{\beta(\tau)-\beta}{R^2+\beta}\right] d\tau \\ & - \int_{0^-}^t \phi_1(t-\tau)\frac{\partial}{\partial\tau}\left\{\frac{\beta(\tau)}{\beta(\tau)-\beta}\log\frac{R^2+\beta(\tau)}{R^2+\beta} + \frac{\beta(\tau)}{\beta}\log\frac{R^2+\beta}{R^2}\right\} d\tau \\ & - \int_{0^-}^t \left[\phi_2(t-\tau) + \frac{1}{2}\phi_3(t-\tau)\right]\frac{\partial}{\partial\tau}\left\{\frac{\beta^2 - 2\beta\beta(\tau)}{[\beta(\tau)-\beta]^2}\log(R^2+\beta)\right. \\ & + \frac{\beta^3(\tau) + \beta^2(\tau)\beta + 2\beta^3 - 4\beta^2\beta(\tau)}{[\beta(\tau)-\beta]^2}\,\frac{1}{R^2+\beta} \\ & \left. + \frac{\beta^2(\tau)}{[\beta(\tau)-\beta]^2}\log[R^2+\beta(\tau)] - \log R^2 + \frac{\beta(\tau)}{R^2}\right\} d\tau. \end{aligned} \tag{4.24}$$

The circumferential stress $\sigma_\theta$ can then be determined by equation (4.23). The hydrostatic pressure function $p(t)$ is found from equation (4.15) which gives the form

$$\begin{aligned} -p(t) = {} & \sigma_r - \left[a + b\left(\frac{\beta}{R^2} - \frac{\beta}{R^2+\beta}\right) + c\left(\frac{2R^2+\beta}{R^2}\right)\right]\frac{R^2}{R^2+\beta} \\ & - 2\int_{0^-}^t \left\{\left[\frac{R^2+\beta}{R^2}\phi_0(t-\tau) + \phi_1(t-\tau) + \frac{R^2}{R^2+\beta}\phi_2(t-\tau)\right]\frac{\partial}{\partial\tau}\left(\frac{R^2}{R^2+\beta(\tau)}\right)\right\} d\tau \\ & + \frac{R^2}{R^2+\beta}\int_{0^-}^t \phi_3(t-\tau)\frac{\partial}{\partial\tau}\left[\frac{R^2+\beta(\tau)}{R^2} + \frac{R^2}{R^2+\beta(\tau)} + 1\right] d\tau. \end{aligned} \tag{4.25}$$

The axial stress $\sigma_z$ can then be calculated by the relation

$$\sigma_z = -p(t)+a+b\left(\frac{\beta}{R^2}-\frac{\beta}{R^2+\beta}\right)+c\left(\frac{\beta}{R^2+\beta}+\frac{R^2+\beta}{R^2}\right)$$
$$+\int_{0^-}^{t}\phi_3(t-\tau)\frac{\partial}{\partial\tau}\left[\frac{R^2+\beta(\tau)}{R^2}+\frac{R^2}{R^2+\beta(\tau)}+1\right]\mathrm{d}\tau. \tag{4.26}$$

The deformations of cylinder can be measured conveniently by using the Lagrangian strain which has the form

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{C}-\mathbf{I}). \tag{4.27}$$

The nonvanishing components for the present problem are: axial strain $E_r$ and circumferential strain $E_\theta$. To express them explicitly in terms of $\beta(t)$, we have

$$E_r = -\frac{\beta}{2(R^2+\beta)} \tag{4.28}$$

$$E_\theta = \frac{\beta}{2R^2}. \tag{4.29}$$

Therefore, the stress and strain components are expressed in terms of two unknown functions $\alpha(t)$ and $\beta(t)$. They should be determined from the given boundary conditions.

The boundary conditions for the problem of a cylinder subjected to an axially symmetrical internal pressure $\pi(t)$ are

$$\sigma_r[R_1, t] = -\pi(t) \tag{4.30}$$

$$\sigma_r[R_2, t] = 0. \tag{4.31}$$

From equation (4.31), $\alpha(t)$ can be found in terms of $\beta(t)$. Then using equation (4.30), an integral equation for solving $\beta(t)$ is found which has the form

$$-\pi(t) = \frac{1}{2}(a+c)\left[\beta\xi(t)+\eta(t)+2\log\frac{R_1}{R_2}\right]$$
$$+\frac{b}{2}\left\{2\log\frac{R_2}{R_1}-\eta(t)+\beta^2\left[\frac{1}{R_2^2(R_1^2+\beta)}-\frac{1}{R_1^2(R_1^2+\beta)}\right]-\frac{R_2^4}{2(R_2^2+\beta)^2}+\frac{R_1^4}{2(R_1^2+\beta)^2}\right\}$$
$$+\int_{0^-}^{t}\left[\phi_0(t-\tau)-\frac{1}{2}\phi_3(t-\tau)\right]\frac{\partial}{\partial\tau}[\eta(\tau)-\eta(t)+(\beta(\tau)-\beta)\xi(t)]\,\mathrm{d}\tau$$
$$+\int_{0^-}^{t}\phi_1(t-\tau)\frac{\partial}{\partial\tau}\left[\frac{\beta(\tau)}{\beta(\tau)-\beta}[\eta(\tau)-\eta(t)]+\frac{\beta(\tau)}{\beta}\left[2\log\frac{R_1}{R_2}+\eta(t)\right]\right]\mathrm{d}\tau$$
$$+\int_{0^-}^{t}\left[\phi_2(t-\tau)+\frac{1}{2}\phi_3(t-\tau)\right]\frac{\partial}{\partial\tau}\left\{\frac{\beta^2-2\beta\beta(\tau)}{[\beta(\tau)-\beta]^2}\eta(t)\right.$$
$$\left.+\frac{\beta^3(\tau)+\beta^2(\tau)\beta+2\beta^3-4\beta^2\beta(\tau)}{[\beta(\tau)-\beta]^2}\xi(t)+\frac{\beta^2(\tau)}{[\beta(\tau)-\beta]^2}\eta(\tau)+2\log\frac{R_1}{R_2}+\frac{(R_1^2-R_2^2)\beta(\tau)}{R_1^2R_2^2}\right\}\mathrm{d}\tau$$

where

$$\xi(t) = \frac{1}{R_2^2+\beta} - \frac{1}{R_1^2+\beta}$$
$$\eta(t) = \log \frac{R_2^2+\beta}{R_1^2+\beta}. \tag{4.32}$$

After the function $\beta(t)$ is found, $\alpha(t)$ and $\sigma_r$ are determined by equations (4.24) and (4.31), respectively, and $\sigma_\theta$ is evaluated from equation (4.23).

In the numerical calculation, the material functions and equilibrium coefficients are assumed the same as those assigned in the uniaxial creep problem. The ratio of the inner and outer radii is chosen to be 0·6. The symmetrical internal pressure is taken to be an instantaneously applied pressure at zero time, or

$$\frac{\pi(t)}{\phi_0(0)} = \pi_0 H(t) \tag{4.33}$$

where $H(t)$ is a Heaviside unit step function. The values of $\pi_0$ varies from $0 \sim 3{\cdot}0$. The unknown function $\beta(t)$ at zero time is determined by a trial-and-error procedure by using the linear solution as the initial trial value. Each successive time step is then determined by a similar iterative method with the value of the previous step as the initial trial value. Integrations are evaluated numerically by a trapezoidal formula. A program in Fortran language was made for the computer CDC 6500. Different time intervals were chosen in the computation. Finally, we found that the time interval $\Delta(t/t_0) = 0{\cdot}05$ was suitable for calculation where $t_0$ is chosen to be 100 sec. Figure 4 shows the radial and circumferential strains at various loading values. The radial strain $E_r$ is approximately following a straight line variation which indicates a nearly linear behavior. However, the circumferential strain $E_\theta$ is concave toward the strain axis and the nonlinearity becomes increasingly important as the loading increases. For strains less than 5 per cent, the sum $E_r+E_\theta$ is close to zero which shows that this particular material can be approximated by a linear theory if the strain level is lower than 5 per cent. The strain distributions in cylinder are shown in Figs. 5 and 6 where $E_r$ and $E_\theta$ are plotted against the dimensionless space variable $R/R_2$ for different times. It shows that the maximum strain occurs at the inner boundary and the value increases with time. Figures 7 and 8 show the stress distributions in cylinder for various times. For a linear analysis, since the stress components $\sigma_r$ and $\sigma_\theta$ are independent of material behavior, they are invariant with respect to time [28]. Therefore, the time variation shown in Figs. 7 and 8 are due to the effect of nonlinearity. As expected from the result of nearly linear variation of $E_r$ shown in Fig. 4, the radial stress does not change appreciably with time. However, the circumferential stress component shows a significant time effect of 15 per cent at $\pi_0 = 2{\cdot}0$. Figures 9 and 10 show the values of stresses at various internal pressures. Again, $\sigma_r$ is essentially linear. $\sigma_\theta$ is concave toward the stress axis and the nonlinearity grows sharply with time.

## 5. CONCLUSIONS

A nonlinear creep problem and a cylinder problem are studied based on the approximate constitutive equation of finite linear viscoelasticity. We have demonstrated that this equation is convenient for stress analysis.

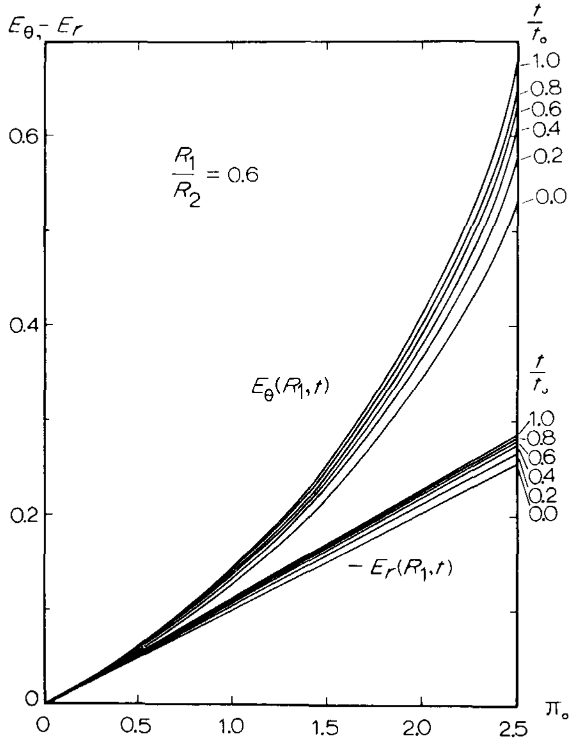


FIG. 4. Radial and circumferential strains vs. loadings.

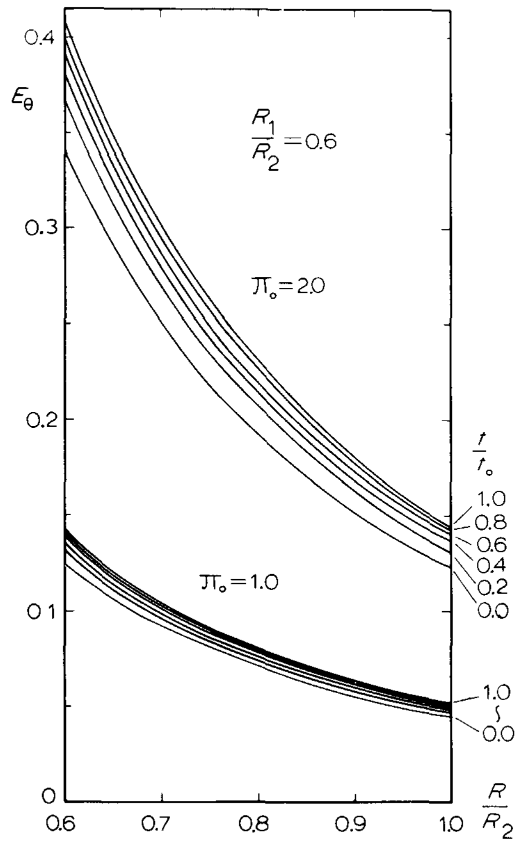


FIG. 5. Circumferential strain distribution in cylinder.

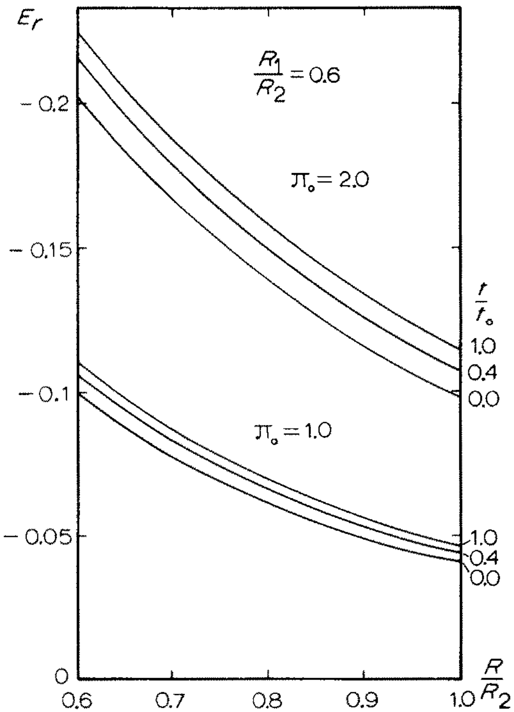


FIG. 6. Radial strain distribution in cylinder.

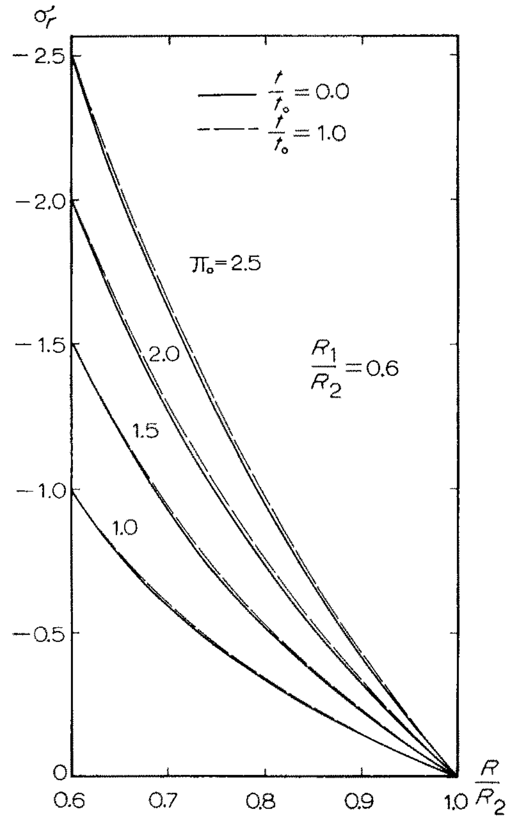


FIG. 7. Radial stress distribution in cylinder.

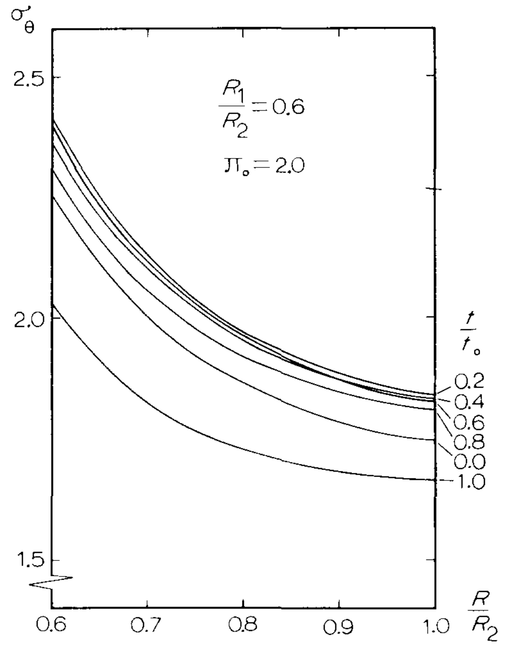


FIG. 8. Circumferential stress in cylinder.

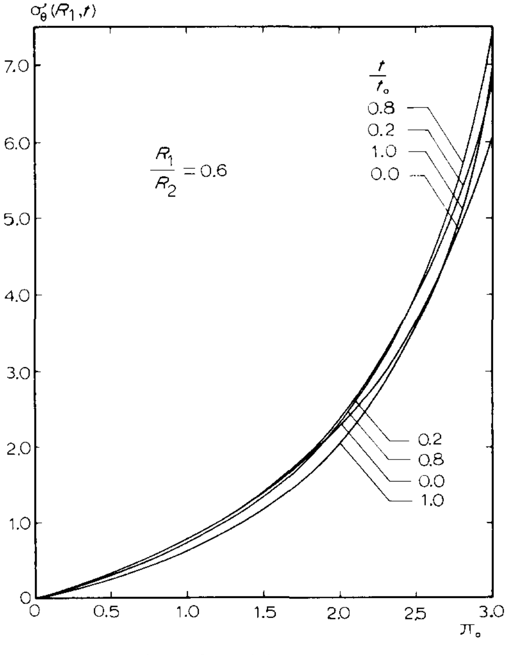


FIG. 9. Circumferential stress vs. loadings.

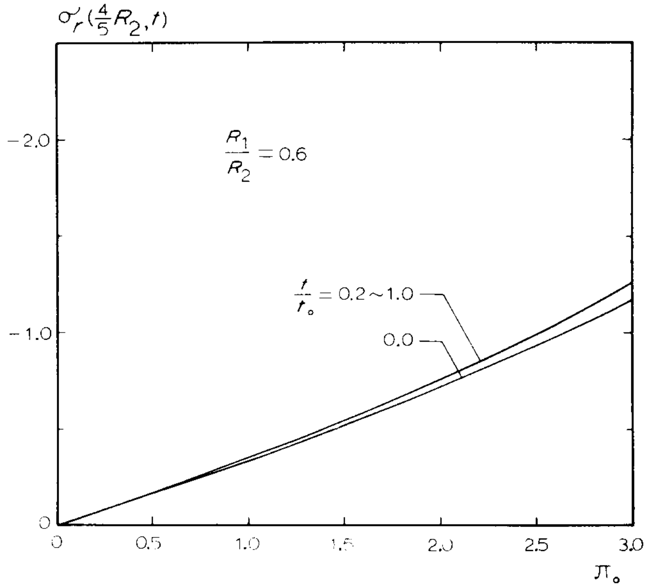


FIG. 10. Radial stress vs. loadings.

It is interesting to note that, in the solution of cylinder problem, the value of $\sigma_\theta(R_1, t)$ in the nonlinear analysis is much higher than the corresponding value in the linear analysis. Hence, for this particular material, an engineering design based on the linear theory could be on the dangerous side. This is in contrast with the previous analyses made by Huang and Lee [30] and Ting [31, 32]. In their numerical calculations based upon the experimental data for uniaxial creep tests on polypropylene, the nonlinear value is lower than the linear solution. The results are different because their constitutive equation is for small finite deformations and the nonlinear effect is contributed primarily by the material nonlinearity. The present investigation is studied for much larger strains, where the geometrical nonlinearity is the dominant factor.

*Acknowledgment*—The support of National Science Foundation under Grant GK-1969 with Purdue University is gratefully acknowledged.

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(*Received* 1 *November* 1971; *revised* 28 *February* 1972)

**Абстракт**—Исследуются нелинейная ползучесть и задача цилиндра, основанные на приближенных конститутивных уравнениях, предложенных Лианисом. Показано, что эти уравнения подходящие для характеристики материалов похожих резине, при конечных деформациях. Используя метод конечных разностей, получаются численные решения определенных с помощью экспериментальных данных для резины стирола бутадиена. Обсуждаются эффекты нелинейности свойств материала.